Lines on the Fermat quintic 3-fold

$Gr(2,5) :=$ 2-planes in k^5 or lines in $\mathbb{P}^4$
$\quad\;\; \ni$
$\quad [L]$

$Sym^5 S^*_{[L]} :=$ degree 5 polynomials on L

The Euler number

$n(Sym^5 S^* \longrightarrow Gr(2,5))$ counts

lines on a quintic 3-fold.

quintic 3-fold $\quad X = \{f = 0\} \subseteq \mathbb{P}^4$

$\uparrow$ degree 5

$\rightsquigarrow$ section $\sigma_f : Gr(2,5) \longrightarrow \mathrm{Sym}^5 S^*$

$$\sigma_f([L]) = f|_L$$

$$\sigma_f([L]) = 0 \iff f|_L = 0 \iff L \subset X$$

Fermat quintic 3-fold

$X = \{ F = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0 \} \subseteq \mathbb{P}^4$

Then $Z := \{ \sigma_F = 0 \} \subseteq Gr(2,5)$ is 1-dim.

Lines on X: u, v coordinates on L, $a^5 + b^5 + c^5 = 0$
ξ = 5th root of unity

$$L = (u : -\xi u : av : bv : cv)$$

$50 = \binom{5}{2} \cdot 5$

So $Z = \bigcup_{50} C$

Albano-Katz (over $k=\mathbb{C}$)

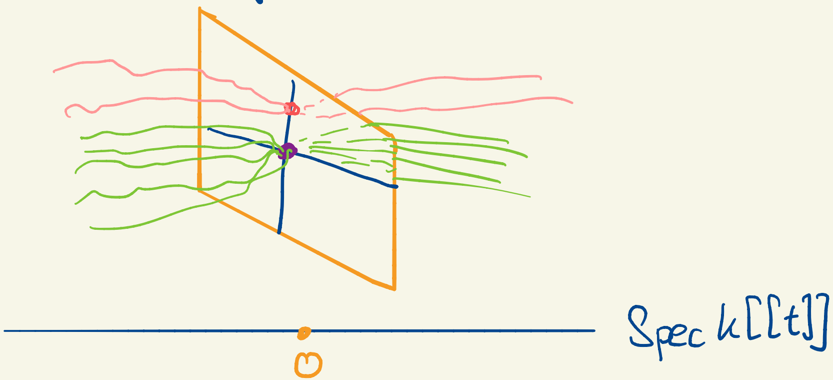
1) There are $\boxed{10}$ lines on each component C that deform with multiplicity $\boxed{2}$.

2) The lines in the intersection of two components deform with multiplicity $\boxed{5}$.

Clemens-Kley: (over $k=\mathbb{C}$)

local analytic structure of Z:

$$\boxed{\frac{\mathbb{C}[x,y]}{(x^3y^2, x^2y^3)}}$$



This still holds for arbitrary fields k of characteristic $\neq 2, 5$ (P.)

$$X_t := \{F + t^5 G = 0\} \subseteq \mathbb{P}^4$$

Take $G = -a x_0^3 x_2^2 - b x_0^2 x_2^3$. $ab \neq 0$

Then $L = (u : -u : v : -v : 0)$ deforms.

$$Z_t = \{\sigma_{F+t^5G} = 0\} \subseteq Gr(2,5)_{k((t))}$$

Locally $Z_t = \operatorname{Spec} \dfrac{k((t))[x,y]}{(\underbrace{x^3y^2 - t^5a}_{f_t},\ \underbrace{x^2y^3 - t^5b}_{g_t})}$

$$x = t\sqrt[5]{\frac{a^3}{b^2}} \qquad y = t\sqrt[5]{\frac{b^3}{a^2}}$$

is a zero of

$$(f_t, g_t) : \mathbb{A}^2_{k((t))} \to \mathbb{A}^2_{k((t))}$$

$$E = \boxed{k((t))(\sqrt[5]{ab})}$$

$$\operatorname{ind}\left(t\sqrt[5]{\frac{a^3}{b^2}},\ t\sqrt[5]{\frac{b^3}{a^2}}\right)$$

$$= \operatorname{Tr}_{E/k((t))}\left(\langle \det \operatorname{jac}(f_t, g_t)\left(t\sqrt[5]{\frac{a^3}{b^2}},\ t\sqrt[5]{\frac{b^3}{a^2}}\right)\rangle\right)$$

$$= \operatorname{Tr}_{E/k((t))}\left(\langle \boxed{5t^4\left(\sqrt[5]{ab}\right)^2} \rangle\right)$$

$$= \operatorname{Tr}_{E/k((t))}\left(\langle \boxed{5} \rangle\right) = \boxed{2\cdot \mathbb{H} + \langle 1 \rangle}$$

$$\in GW(k((t)))$$

1) So $L = (u : -u : v : -v : 0)$

contributes $2\mathbb{H} + \langle 1 \rangle$ to $n(\mathrm{Sym}^5 S^*)$.

2) $L = (u : -\zeta u : v : -\zeta' v : 0)$

contributes $\mathrm{Tr}_{k(\zeta)/k}(2\mathbb{H} + \langle 1 \rangle)$

$= 8\mathbb{H} + \mathbb{H} + \langle 1 \rangle + \langle -5 \rangle$

3) The multiplicity 2 lines all

contribute an $\mathbb{H}$.

So $n(\mathrm{Sym}^5 S^*) = 15 \cdot (2\mathbb{H} + \langle 1 \rangle)$

$$+ 90 \cdot (9\mathbb{H} + \langle 1 \rangle + \langle -5 \rangle)$$

$$+ 50 \cdot 10 \cdot \mathbb{H}$$

$$= (30 + 90 \cdot 10 + 500) \cdot \mathbb{H} \quad \in GW(k)$$

char $k \neq 5$

$$+ 15\langle 1 \rangle$$

$$= 1430\,\mathbb{H} + 15\langle 1 \rangle$$

rk: 2875

sgn: 15